

# DESUSPENSIONS OF $S^1 \wedge (\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$

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ABSTRACT. We use the Galois action on  $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$  to show that the homotopy equivalence  $S^1 \wedge (\mathbb{G}_{m, \mathbb{Q}} \vee \mathbb{G}_{m, \mathbb{Q}}) \cong S^1 \wedge (\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$  coming from purity does not desuspend to a map  $\mathbb{G}_{m, \mathbb{Q}} \vee \mathbb{G}_{m, \mathbb{Q}} \rightarrow \mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$ .

## 1. INTRODUCTION

The étale fundamental group of the scheme  $\mathbb{P}^1 - \{0, 1, \infty\}$  contains interesting arithmetic [Del89] [Iha86]. By viewing schemes as objects in the  $\mathbb{A}^1$ -homotopy category of Morel-Voevodsky [MV99], we may form the simplicial suspension  $\Sigma X = S^1 \wedge X$  of a pointed scheme  $X$ , and the wedge product of two pointed schemes. After one simplicial suspension,  $\mathbb{P}^1 - \{0, 1, \infty\}$  and the wedge  $\mathbb{G}_m \vee \mathbb{G}_m$  of two copies of  $\mathbb{G}_m$  become canonically  $\mathbb{A}^1$ -equivalent by the purity theorem [MV99, Theorem 2.23], and this is given in Proposition 3.1. This paper uses calculations of Anderson, Coleman, Ihara and collaborators [And89] [Col89] [Iha91] [IKY87] on the étale fundamental group of  $\mathbb{P}^1 - \{0, 1, \infty\}$  to show that this  $\mathbb{A}^1$ -equivalence does not desuspend, in the sense that there does not exist a map  $\mathbb{G}_{m, \mathbb{Q}} \vee \mathbb{G}_{m, \mathbb{Q}} \rightarrow \mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$  whose suspension is equivalent to the map coming from purity. This can be summarized by the statement that the Galois action on the étale fundamental group of  $\mathbb{P}^1 - \{0, 1, \infty\}$  is an obstruction to desuspension.

There are topological obstructions to desuspension coming from James-Hopf maps, and they generalize to the setting of  $\mathbb{A}^1$ -homotopy theory [WW14] [AFWW15] as was known to Morel. They do not a priori have a relationship with the Galois action on  $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$ . This paper is motivated by the contrast between the systematic tools from algebraic topology available to obstruct desuspension and the arithmetic of  $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\})$  which shows that such a desuspension does not exist.

The results of this paper are as follows. Let  $\mathbf{Sm}_k$  denote the full subcategory of finite type schemes over a characteristic 0 field  $k$  whose objects are smooth schemes. Let  $\mathbf{sPre}(\mathbf{Sm}_k)$  denote presheaves of simplicial sets on  $\mathbf{Sm}_k$ .  $\mathbf{sPre}(\mathbf{Sm}_k)$  has the structure of a simplicial model category in several ways. Let  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  denote the homotopy category of the  $\mathbb{A}^1$ -local, projective étale (respectively Nisnevich) model structure on  $\mathbf{sPre}(\mathbf{Sm}_k)$ . The results of this paper hold with either the Nisnevich or étale Grothendieck topology, and we will use the same notation for either.  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  is formed by formally inverting  $\mathbb{A}^1$ -weak equivalences and local étale (respectively Nisnevich) equivalences. In particular, in the injective Nisnevich case,  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  is the  $\mathbb{A}^1$ -homotopy category of [MV99].

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In Section 3, we give the canonical isomorphism in  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  discussed above between the unreduced simplicial suspension of  $\mathbb{P}_k^1 - \{0, 1, \infty\}$  and the simplicial suspension  $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) = S^1 \wedge (\mathbb{G}_m \vee \mathbb{G}_m)$ . The reduced and unreduced simplicial suspensions are canonically isomorphic in  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ , although to form the reduced simplicial suspension, a base point is required. Let  $\wp : \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \rightarrow \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$  denote any of the maps in  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  resulting from choosing a base point in  $\mathbb{P}_k^1 - \{0, 1, \infty\}$ .

**Theorem 1.** *Let  $k$  be a finite extension of  $\mathbb{Q}$  not containing a square root of 2. There is no morphism  $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k} \rightarrow \mathbb{P}_k^1 - \{0, 1, \infty\}$  in  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  whose simplicial suspension is  $\wp$ .*

Theorem 1 is proved as Theorem 4.2. The proof uses an étale realization, following ideas of Artin-Mazur, Friedlander, Isaksen, Quick, and Schmidt. The needed results are collected in Section 2.

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## 2. EQUIVARIANT ÉTALE HOMOTOPY TYPE

Let  $\bar{k}$  be an algebraic closure of  $k$ , and  $G = \mathrm{Gal}(\bar{k}/k)$  denote the absolute Galois group of  $k$ . Let  $\mathbb{Z}^\wedge$  denote the profinite completion of  $\mathbb{Z}$ . Let  $\chi : G \rightarrow (\mathbb{Z}^\wedge)^*$  denote the cyclotomic character. Let  $\mathbb{Z}^\wedge(\mathbf{n})$  denote  $\mathbb{Z}^\wedge$  with the continuous  $G$ -action where  $g$  in  $G$  acts by multiplication by  $\chi^n(g)$ . For a pro-group  $J = \{J_\alpha\}_{\alpha \in A}$ , let  $\mathbf{BP}$  denote the pro-simplicial set  $\{BJ_\alpha\}_{\alpha \in A}$  given as the inverse system of the classifying spaces  $BJ_\alpha$  of the groups  $J_\alpha$ . For a group  $J$ , let  $\mathrm{pro} - J^\wedge$  denote the pro-group given as the inverse system of the finite quotients of  $J$ .

Let  $\mathrm{Et} : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{pro} - \mathbf{sSet}$  denote the étale homotopy type of [Isa04, Definition 1]. This étale homotopy type is built using the étale homotopy and topological types of [AM86] and [Fri82]. An alternate extension of the étale homotopy type to the  $\mathbb{A}^1$ -homotopy category of schemes was constructed independently by Alexander Schmidt [Sch03] [Sch12]. By [Isa04, Theorem 2],  $\mathrm{Et}$  is a left Quillen functor with respect to the étale or Nisnevich local projective model structure on  $\mathbf{sPre}(\mathbf{Sm}_k)$  and the model structure on  $\mathbf{pro} - \mathbf{sSet}$  given in [Isa01]. This model structure on  $\mathbf{pro} - \mathbf{sSet}$  is such that weak equivalences are the maps  $f : X \rightarrow Y$  inducing an isomorphism of pro-sets  $\pi_0(X) \rightarrow \pi_0(Y)$  and inducing an isomorphism from the local systems associated to  $\pi_n(X)$  to the pull-back of the local system on  $Y$  coming from  $\pi_n(Y)$ . The cofibrations are maps isomorphic to levelwise cofibrations. See [Isa04, Definition 6.1, 6.2]. The proof that  $\mathrm{Et}$  is a left Quillen functor uses [Dug01, Prop 2.3]. Let  $\mathrm{LEt} : \mathrm{ho} \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathrm{ho} \mathbf{pro} - \mathbf{sSet}$  denote the corresponding homotopy invariant derived functor. By [Isa04, Corollary 4], the functor  $\mathrm{LEt}$  agrees with the étale homotopy type for a scheme in  $\mathbf{Sm}_k$ . The standard calculation of the étale homotopy type of  $\mathrm{Spec} k$  then gives  $\mathrm{LEt}(\mathrm{Spec} k) \cong \mathbf{B}(\mathrm{pro} - G^\wedge)$ .

**2.1. Fundamental groupoids.** Let  $X$  be a scheme over  $k$ . Let  $\Pi_1^{\mathrm{ét}} X$  denote the étale fundamental groupoid of  $X$  whose objects are geometric points and morphisms are étale

paths, i.e., natural transformations between the associated fiber functors [SGAI, V 7]. The morphisms are topologized with the natural profinite topology.

For a simplicial set  $X$ , let  $\Pi_1$  denote the fundamental groupoid. For a pro-simplicial set  $X = \{X_\alpha\}_{\alpha \in A}$ , let  $\Pi_1 X$  denote the pro-groupoid  $\{\Pi_1 X_\alpha\}_{\alpha \in A}$ .

**2.2. Base points and fundamental groups.** Given a map  $* \rightarrow X = \{X_\alpha\}_{\alpha \in A}$  in **pro** – **sSet**, each simplicial set  $X_\alpha$  has a base-point coming from the definition of morphisms

$$\mathrm{Hom}(*, X) = \varprojlim_{\alpha \in A} \varinjlim \mathrm{Hom}(*, X_\alpha) \cong \varprojlim_{\alpha \in A} \mathrm{Hom}(*, X_\alpha)$$

in the pro-category. Thus there is a distinguished object in each  $\Pi_1 X_\alpha$ . The endomorphisms of this object fit into a pro-group, defined to be the fundamental group  $\pi_1(X)$  of  $* \rightarrow X$ .

For  $X$  in **sPre(Sm<sub>k</sub>)**, use the notation  $\Pi_1(X)$  for  $\Pi_1 \mathrm{LEt} X$ . The étale homotopy type  $\mathrm{Et}$  takes a scheme  $X$  equipped with a geometric point  $\mathrm{Spec} \Omega \rightarrow X$  to a pointed pro-simplicial set because  $\mathrm{Et}(\mathrm{Spec} \Omega) \cong *$  for  $\Omega$  an algebraically closed field. The resulting  $\pi_1$  is independent of the choice of isomorphism  $\mathrm{Et}(\mathrm{Spec} \Omega) \cong *$ .

Let  $\bar{x} : \mathrm{Spec} \Omega \rightarrow X$  be a geometric point. Replace  $\Omega$  by the subfield of  $\Omega$  given by the algebraic closure of the residue field of the point of  $X$  in the image of  $\bar{x}$ . This replacement has finite transcendence degree, and therefore  $\mathrm{Spec} \Omega$  is an essentially smooth  $k$ -scheme in the sense of [Mor12, vi], i.e. a Noetherian scheme which is an inverse limit of a left filtering system  $\{\Omega_\alpha\}_{\alpha \in A}$  of smooth  $k$ -schemes with étale affine transition morphisms. Since  $X$  is assumed to be finite type over  $k$ , the map  $\bar{x}$  is determined by the images of finitely many functions on an open subset of  $X$ , and thus determines an element of  $\varinjlim X(\Omega_\alpha)$ . As in [Mor12], given  $X \in \mathbf{sPre}(\mathbf{Sm}_k)$  and an essentially smooth  $k$ -scheme  $\{Y_\alpha\}_{\alpha \in A}$ , define  $X(Y) = \varinjlim X(Y_\alpha)$ , and call  $X(Y)$  the set of  $Y$  points of  $X$ . For  $X$  in **sPre(Sm<sub>k</sub>)**, a *geometric point* of  $X$  indicates an element of  $X(\mathrm{Spec} \Omega)$ , where  $\Omega$  is an algebraically closed field of finite transcendence degree over  $k$ . Note that for  $X \in \mathbf{sPre}(\mathbf{Sm}_k)$ , a geometric point  $\bar{x} \in X(\mathrm{Spec} \Omega)$  induces a map  $\mathrm{Et}(\mathrm{Spec} \Omega) \rightarrow \mathrm{LEt} X$ .

Let  $k\text{--}\mathbf{Sm}^+$  denote the following category. The objects of  $k\text{--}\mathbf{Sm}^+$  are pairs  $(X, \bar{x})$ , where  $X$  is a smooth  $k$ -scheme and  $\bar{x}$  is a geometric point of  $X$  equipped with a path between its image under  $X \rightarrow \mathrm{Spec} k$  and the geometric point  $\mathrm{Spec} \bar{k} \rightarrow \mathrm{Spec} k$  of  $k$ . The morphisms  $(X, \bar{x}) \rightarrow (Y, \bar{y})$  of  $k\text{--}\mathbf{Sm}^+$  are the morphisms  $X \rightarrow Y$  in  $k\text{--}\mathbf{Sm}$ . There is no requirement that  $\bar{x}$  is taken to  $\bar{y}$ . Let  $k\text{--}\mathbf{Sm}_c^+$  denote the full subcategory of  $k\text{--}\mathbf{Sm}^+$  of objects such that  $X$  is connected. Let  $\mathbf{Grp}_G^{\mathrm{out}}$  denote the category of topological groups over  $G$  and outer homomorphisms, i.e., the objects of  $\mathbf{Grp}_G^{\mathrm{out}}$  are morphisms  $\pi \rightarrow G$  and the morphisms from  $\pi \rightarrow G$  to  $\pi' \rightarrow G$  is the set of equivalence classes of morphisms  $\pi \rightarrow \pi'$  such that the two morphisms  $\pi \rightarrow G$  coming from the diagram

$$\begin{array}{ccc} \pi & \xrightarrow{\quad} & \pi' \\ & \searrow & \swarrow \\ & G & \end{array}$$

differ by an inner automorphism of  $G$  and where two such morphisms  $f, f' : \pi \rightarrow \pi'$  are considered equivalent if there exists  $\gamma \in \pi'$  such that  $f'(x) = \gamma f(x) \gamma^{-1}$  for all  $x$  in  $\pi$ .

Given a morphism  $\Pi \rightarrow \Pi'$  of pro-groupoids and commutative diagram

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi' \\ \uparrow & & \uparrow \\ * & \longrightarrow & * \end{array},$$

there is an associated morphism of fundamental pro-groups  $\pi \rightarrow \pi'$ , where  $\pi$  is the endomorphisms in  $\Pi$  of the distinguished object and similarly for  $\pi'$ . Given two maps  $x_1, x_2 : * \rightarrow \Pi'$  and a choice of morphism from  $x_1$  to  $x_2$  we obtain an isomorphism between the fundamental group based at  $x_1$  and the fundamental group based at  $x_2$ . A different choice of path changes the isomorphism by an inner isomorphism. To a map between objects of  $k - \mathbf{Sm}_c^+$ , we may therefore associate an outer homomorphism. We claim that this defines a functor  $\pi_1^{\text{ét}} : k - \mathbf{Sm}_c^+ \rightarrow \mathbf{Grp}_G^{\text{out}}$ . To see this, note that for the object  $(X, \bar{x})$  of  $k - \mathbf{Sm}_c^+$ , the path between the image of  $\bar{x}$  under  $X \rightarrow \text{Spec } k$  and the geometric point  $\text{Spec } \bar{k} \rightarrow \text{Spec } k$  produces a morphism  $\pi_1^{\text{ét}}(X, \bar{x}) \rightarrow G$ . Given  $(X, \bar{x}) \rightarrow (Y, \bar{y})$ , the induced outer homomorphism  $\pi_1^{\text{ét}}(X, \bar{x}) \rightarrow \pi_1^{\text{ét}}(Y, \bar{y})$  respects the maps to  $G$  up to inner automorphism because  $X \rightarrow Y$  respects the maps to  $\text{Spec } k$ . This shows that  $\pi_1^{\text{ét}}$  determines the claimed functor.

We need a fundamental group on pointed objects of  $\mathbf{sPre}(\mathbf{Sm}_k)$  with similar functoriality properties, so we introduce notation in this context analogous to the above. Let  $\mathbf{sPre}(\mathbf{Sm}_k)^+$  denote the category whose objects are pairs  $(X, \bar{x})$ , where  $X$  is in  $\mathbf{sPre}(\mathbf{Sm}_k)$  and  $\bar{x}$  is a geometric point of  $X$  whose image in the set of geometric points of  $\text{Spec } k$  has a chosen path to  $\text{Spec } \bar{k} \rightarrow \text{Spec } k$ , and whose morphisms  $(X, \bar{x}) \rightarrow (Y, \bar{y})$  are the morphisms  $X \rightarrow Y$  in  $\mathbf{sPre}(\mathbf{Sm}_k)$ . There is again no requirement that  $\bar{x}$  is taken to  $\bar{y}$ . Define  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)^+$  similarly, i.e., the morphisms  $(X, \bar{x}) \rightarrow (Y, \bar{y})$  are the morphisms  $X \rightarrow Y$  in  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ . Let  $\mathbf{sPre}(\mathbf{Sm}_k)_c^+$  denote the full-subcategory of  $\mathbf{sPre}(\mathbf{Sm}_k)^+$  on objects such that  $\text{LEt } X$  is connected. Similarly define  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_c^+$  to be the full-subcategory of  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)^+$  on objects such that  $\text{LEt } X$  is connected. Let  $\text{pro} - \mathbf{Grp}_{\text{pro} - G^\wedge}^{\text{out}}$  denote the category of pro-groups over  $\text{pro} - G^\wedge$  and outer homomorphisms.

For  $(X, \bar{x})$  in  $\mathbf{sPre}(\mathbf{Sm}_k)^+$ , define  $\pi_1(X, \bar{x} : \text{Spec } \Omega \rightarrow X)$  to be  $\pi_1$  of the pointed pro-simplicial set  $* \cong \text{Et}(\text{Spec } \Omega) \rightarrow \text{LEt } X$ . By the same argument as above,  $\pi_1$  defines a functor  $\pi_1 : \mathbf{sPre}(\mathbf{Sm}_k)_c^+ \rightarrow \text{pro} - \mathbf{Grp}_{\text{pro} - G^\wedge}^{\text{out}}$ .

### 2.3. Homotopy invariant functors.

**Proposition 2.1.** *The functor  $\pi_1 : \mathbf{sPre}(\mathbf{Sm}_k)_c^+ \rightarrow \text{pro} - \mathbf{Grp}_{\text{pro} - G^\wedge}^{\text{out}}$  factors through  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_c^+$ . Furthermore, the diagram*

$$\begin{array}{ccc} k - \mathbf{Sm}^+ & \xrightarrow{\pi_1^{\text{ét}}} & \mathbf{Grp}_G^{\text{out}} \\ \downarrow & & \uparrow \text{lim} \\ \text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_c^+ & \xrightarrow{\pi_1} & \text{pro} - \mathbf{Grp}_{\text{pro} - G^\wedge}^{\text{out}} \end{array}$$

*commutes up to isomorphism.*

*Proof.* For a scheme  $X$ , let  $\mathcal{X}$  in  $\mathbf{sPre}(\mathbf{Sm}_k)$  denote the corresponding sheaf. There is a natural isomorphism  $\pi_1^{\text{ét}}(X) \cong \varprojlim \pi_1 \text{LEt } \mathcal{X}$  for every smooth scheme  $X$  over  $k$  equipped

with a geometric point because both sides classify finite étale covers of  $X$ . For the left hand side, this is immediate. For the right hand side, this follows from [Fri82, Prop 5.6] and [AM86, 11.1].

To show that  $\pi_1$  factors through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_c^+$ , it suffices to show that the functor  $\Pi_1$  from spaces to pro-groupoids factors through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ . Since  $\Pi_1 = \Pi_1 \mathrm{LEt}$ , we know that  $\Pi_1$  factors through the homotopy category of the étale (respectively Nisnevich) local projective model structure. To show the factorization through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ , it is thus sufficient to show that  $X \times \mathbb{A}^1 \rightarrow X$  is sent to an isomorphism for all schemes  $X$ . This follows from the analogous claim on étale fundamental groups, which is true in characteristic 0. (One can see that  $\pi_1^{\mathrm{ét}}(X \times \mathbb{A}^1) \rightarrow \pi_1^{\mathrm{ét}} X$  is an isomorphism in characteristic 0 by combining [SGAI, IX Théorème 6.1] with the analogous result over  $\bar{k}$ . Over  $\bar{k}$ , the map is an isomorphism by invariance of  $\pi_1^{\mathrm{ét}}$  under algebraically closed extensions of fields [SGAI, XIII Proposition 4.6] and comparison with the topological fundamental group [SGAI, XII Corollaire 5.2].)  $\square$

**Example 2.2.** We compute  $\pi_1(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *) \rightarrow G$ . The map  $* \rightarrow \mathbb{G}_{m,k}$  corresponding to the point 1 is a flasque cofibration because it is the push-out product of itself and  $\partial \Delta^0 \rightarrow \Delta^0$ . See [Isa05, Definition 3.2]. Since representable presheaves are projective cofibrant,  $*$  and  $\mathbb{G}_{m,k}$  are projective cofibrant, whence also flasque cofibrant. It follows that  $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$  is a homotopy colimit in the flasque model structure. Since Et is a left Quillen functor on the Nisnevich (or étale) local flasque model structure [Qui08, Theorem 3.4] and since there is a weak equivalence between Et derived with respect to the local flasque model structure, and LEt, which denotes Et derived with respect to the projective local model structure, it follows that

$$\begin{array}{ccc} \mathrm{LEt}(\mathrm{Spec} k) & \longrightarrow & \mathrm{LEt}(\mathbb{G}_{m,k}) \\ \downarrow & & \downarrow \\ \mathrm{LEt}(\mathbb{G}_{m,k}) & \longrightarrow & \mathrm{LEt}(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \end{array}$$

is a homotopy push-out square.

Let  $\{\langle x, y | x^n = 1 = y^n \rangle\}_n$  denote the pro-group given as the inverse system over  $n$  of the free product of  $\mathbb{Z}/n$  with  $\mathbb{Z}/n$  and transition maps induced by quotient maps  $\mathbb{Z}/(nm) \rightarrow \mathbb{Z}/n$ . Let  $G$  act on  $\langle x, y | x^n = 1 = y^n \rangle$  by

$$gx = x^{x(g)} \quad gy = y^{x(g)}.$$

Let  $I$  denote the directed set consisting of pairs  $(n, H)$  with  $n$  a positive integer and  $H$  a finite quotient of  $G$  which acts on  $k(\mu_n)$ , i.e.,  $H$  is such that the fixed field of  $\mathrm{Ker}(G \rightarrow H)$  contains the  $n$ th roots of unity in  $k$ .  $I$  is defined so that there is a map  $(n, H) \rightarrow (n', H')$  exactly when  $H'$  is a quotient of  $H$  and  $n'$  is a quotient of  $n$ .

We claim that  $\mathrm{LEt}(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \rightarrow \mathrm{LEt} \mathrm{Spec} k$  can be identified with

$$\{B(\langle x, y | x^n = 1 = y^n \rangle \rtimes H)\}_{(n,H) \in I} \rightarrow B \mathrm{pro} G^{\wedge}.$$

Since  $\mathrm{LEt} \mathbb{G}_m$  is the étale topological type, the map  $\mathrm{LEt} \mathbb{G}_m \rightarrow \mathrm{LEt} \mathrm{Spec} k$  can be identified with  $\mathrm{B} \mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge \rightarrow \mathrm{B} \mathrm{pro} G$ . It thus suffices to show that

$$(1) \quad \begin{array}{ccc} \mathrm{B} \mathrm{pro} G & \longrightarrow & \mathrm{B} \mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge \\ \downarrow & & \downarrow \\ \mathrm{B} \mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge & \longrightarrow & \{\mathrm{B}\langle x, y | x^n = 1 = y^n \rangle \rtimes H\}_{(n,H) \in I} \end{array}$$

is a homotopy push-out square. Since  $\mathrm{B} \mathrm{pro} G \rightarrow \mathrm{B} \mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge$  is a level-wise section of a level-wise fibration of simplicial sets, it is isomorphic to a level-wise monomorphism and is therefore a cofibration. Thus it suffices to show that (1) is a push-out.

To see this, let  $D$  and  $F$  be finite groups, with actions of a finite group  $C$ . By Van-Kampen's theorem,

$$(2) \quad \begin{array}{ccc} * & \longrightarrow & BF \\ \downarrow & & \downarrow \\ BD & \longrightarrow & B(D * F) \end{array}$$

is a push-out and a homotopy push-out, where  $D * F$  denotes the free product of  $D$  and  $F$ . Let  $EC$  denote a universal cover of  $BC$ . Applying  $(-) \times_G EC$  to (2) produces another push-out and homotopy push-out. It follows that (1) is a push-out, as claimed.

Thus  $\pi_1(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *) \rightarrow G$  can be identified with the map

$$\{\langle x, y | x^n = 1 = y^n \rangle \rtimes H\}_{(n,H) \in I} \rightarrow \mathrm{pro} - G^\wedge.$$

Define  $\pi'$  to be the free profinite group on two generators  $\pi' = \langle x, y \rangle^\wedge$ , and let  $G$  act on  $\pi'$  by

$$(3) \quad gx = x^{x(g)} \quad gy = y^{x(g)}.$$

**Lemma 2.3.** Any morphism in  $\mathrm{pro} - \mathbf{Grp}_{\mathrm{pro} - G^\wedge}^{\mathrm{out}}$  from  $\{\langle x, y | x^n = 1 = y^n \rangle \rtimes H\}_{(n,H) \in I}$  to an inverse system of finite groups factors through  $\mathrm{pro} - (\pi' \rtimes G)^\wedge$ .

*Proof.* Let  $\{J_\alpha\}_{\alpha \in A}$  be a pro-group with each  $J_\alpha$  finite, and suppose  $\{J_\alpha\}_{\alpha \in A}$  is equipped with a map  $\{J_\alpha\}_{\alpha \in A} \rightarrow \mathrm{pro} - G^\wedge$ . Any morphism in  $\mathrm{pro} - \mathbf{Grp}_{\mathrm{pro} - G^\wedge}^{\mathrm{out}}$  from  $\{\langle x, y | x^n = 1 = y^n \rangle \rtimes H\}_{(n,H) \in I}$  to  $\{J_\alpha\}_{\alpha \in A}$  is represented by a morphism of pro-groups

$$\{\langle x, y | x^n = 1 = y^n \rangle \rtimes H\}_{(n,H) \in I} \rightarrow \{J_\alpha\}_{\alpha \in A}.$$

Such a morphism is an element of

$$(4) \quad \varprojlim_\alpha \varinjlim_{(n,H)} \mathrm{Hom}(\langle x, y | x^n = 1 = y^n \rangle \rtimes H, J_\alpha).$$

Since  $J_\alpha$  is finite, this set is in natural bijection with

$$\varprojlim_\alpha \varinjlim_{(n,H)} \mathrm{Hom}((\langle x, y | x^n = 1 = y^n \rangle \rtimes H)^\wedge, J_\alpha).$$

Since there is a map  $\langle x, y \rangle \rightarrow \langle x, y | x^n = 1 = y^n \rangle \rtimes H$  sending  $x$  to  $x \rtimes 1$  and  $y$  to  $y \rtimes 1$ , there is an induced map  $\pi' \rightarrow (\langle x, y | x^n = 1 = y^n \rangle \rtimes H)^\wedge$ . Since this map is equivariant with respect to the quotient  $G \rightarrow H$ , there is an induced map  $\pi' \rtimes G \rightarrow (\langle x, y | x^n = 1 = y^n \rangle \rtimes H)^\wedge$ .



By checking compatibility with the transition maps, it follows that the set of morphisms (4) is in natural bijection with

$$\varprojlim_{\alpha} \mathrm{Hom}(\pi' \rtimes G, J_{\alpha}).$$

□

Let  $H^i(-, \mathbb{Z}/n) : \mathbf{pro-sSet} \rightarrow \mathbf{Ab}$  denote the functor which takes a pro-simplicial set  $\{X_{\alpha}\}_{\alpha \in I}$  to the abelian group  $\mathrm{colim}_{\alpha \in I} H^i(X_{\alpha}, \mathbb{Z}/n)$ , cf. [Fri82, §5]. By [Isa01, Proposition 18.4],  $H^i(-, \mathbb{Z}/n)$  passes to the homotopy category and determines a functor

$$H^i(-, \mathbb{Z}/n) : \mathrm{ho-pro-sSet} \rightarrow \mathbf{Ab}.$$

Let  $H_{\mathrm{\acute{e}t}}^i(-, \mathbb{Z}/n)$  denote the usual étale cohomology groups of a scheme with coefficients in  $\mathbb{Z}/n$ .

**Proposition 2.4.**  $H_{\mathrm{\acute{e}t}}^i(-, \mathbb{Z}/n) : \mathbf{Sm}_k \rightarrow \mathrm{pro-Ab}$  factors through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ .

*Proof.* By [Isa04, Corollary 4],  $\mathrm{LEt} X$  is the étale topological type of [Fri82]. Thus by [Fri82, Proposition 5.9],  $H^i(\mathrm{LEt} X, \mathbb{Z}/n)$  is naturally isomorphic to the étale cohomology  $H_{\mathrm{\acute{e}t}}^i(X, \mathbb{Z}/n)$ . Thus it suffices to show that

$$H^i(\mathrm{LEt}(-), \mathbb{Z}/n) : \mathrm{ho-sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{Ab}$$

factors through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ . Since the  $\mathbb{A}^1$ -model structure is obtained by left Bousfield localization at the maps  $X \times \mathbb{A}^1 \rightarrow X$  for every scheme  $X$ , it suffices to show that  $\mathrm{LEt}$  takes  $X \times \mathbb{A}^1 \rightarrow X$  to an isomorphism of abelian groups. This is true by [Mil80, VI Corollary 4.20]. □

Let  $H^i(-, \mathbb{Z}/n)$  also denote the functor  $H^i(-, \mathbb{Z}/n) : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{Ab}$  given by  $H^i(\mathrm{LEt}(-), \mathbb{Z}/n)$ . As in Proposition 2.4,  $H^i(-, \mathbb{Z}/n)$  factors through  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ .

**Proposition 2.5.** *There is a natural isomorphism of functors  $H^i(-, \mathbb{Z}/n) \cong H^{i+1}(\Sigma(-), \mathbb{Z}/n)$ .*

*Proof.* Let  $X$  be an object of  $\mathbf{sPre}(\mathbf{Sm}_k)_*$ . Since left derived functors commute with homotopy colimits,

$$\begin{array}{ccc} \mathrm{LEt} X & \longrightarrow & \mathrm{LEt} * \\ \downarrow & & \downarrow \\ \mathrm{LEt} * & \longrightarrow & \mathrm{LEt} \Sigma X \end{array}$$

is a push-out square in the model structure of [Isa01].

In the model structure of [Isa01], the cofibrations are isomorphic to levelwise cofibrations of systems of simplicial sets of the same shape. Also, levelwise homotopy equivalences are weak equivalences. It follows that

$$\begin{array}{ccc} \{X_{\alpha}\}_{\alpha \in A} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \{\Sigma X_{\alpha}\}_{\alpha \in A} \end{array}$$

is a homotopy push-out. In particular, letting  $\{X_\alpha\}_{\alpha \in A} = \text{LEt } X$ , we have that  $\text{LEt } \Sigma X \cong \{\Sigma X_\alpha\}_{\alpha \in A}$ .

The proposition then follows from the fact that in ordinary cohomology of simplicial sets, we have  $H^{i+1}(\Sigma A_\alpha, \mathbb{Z}/n) \cong H^i(A_\alpha, \mathbb{Z}/n)$ .  $\square$

**Proposition 2.6.** *There is a natural isomorphism of functors*

$$H^1(-, \mathbb{Z}/n) \cong \text{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathbf{sPre}(\mathbf{Sm}_k)_c^+ \rightarrow \mathbf{Ab}.$$

*Proof.* The claim is equivalent to exhibiting a natural isomorphism

$$H^1(\text{LEt}(-), \mathbb{Z}/n) \cong \text{Hom}(\pi_1 \text{LEt}(-), \mathbb{Z}/n).$$

There is natural isomorphism a natural isomorphism

$$H^1(-, \mathbb{Z}/n) \cong \text{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathbf{ho sSet} \rightarrow \mathbf{Ab}.$$

This induces a natural isomorphism

$$H^1(-, \mathbb{Z}/n) \cong \text{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathbf{ho pro-sSet} \rightarrow \mathbf{Ab},$$

where  $\text{Hom}$  is the homomorphisms in the category of pro-groups. The desired natural isomorphism is obtained by pulling back by  $\text{LEt}$ .  $\square$

### 3. STABLE ISOMORPHISM $\mathbb{P}_k^1 - \{0, 1, \infty\} \cong \mathbb{G}_m \vee \mathbb{G}_m$

Recall that the smash product  $X \wedge Y$  of two pointed spaces  $X$  and  $Y$  is  $X \wedge Y = X \times Y / (* \times Y \cup X \times *)$ , and that the wedge product  $X \vee Y$  is the disjoint union with the two base points identified. These formulas hold sectionwise for simplicial presheaves, e.g.  $(X \vee Y)(U) = X(U) \vee Y(U)$ . The simplicial suspension  $\Sigma X$  of  $X$  in  $\mathbf{sPre}(\mathbf{Sm}_k)$  is  $\Sigma X = S^1 \wedge X$ . Let  $S$  denote the unreduced simplicial suspension,  $SX = I \times X / \sim$ , where  $I$  denotes the standard 1-simplex, and  $\sim$  denotes the equivalence relation defined  $0 \times X \sim *_0$  and  $1 \times X \sim *_1$ , where  $*_0$  and  $*_1$  are two copies of the terminal object.

**Proposition 3.1.** *There is a canonical isomorphism  $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \rightarrow S(\mathbb{P}_k^1 - \{0, 1, \infty\})$  in  $\mathbf{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  which sends  $*_0$  to the base point.*

*Proof.* Let  $i : Z \rightarrow \mathbb{A}_k^1$  be the reduced closed subscheme corresponding to the closed set  $\{0, 1\}$ . Note that  $\mathbb{A}_k^1 - i(Z) \cong \mathbb{P}_k^1 - \{0, 1, \infty\}$  is an isomorphism of schemes. Let  $\mathcal{N}(i) \rightarrow Z$  denote the normal bundle to  $i$ , and let  $\text{Th}(\mathcal{N}(i))$  denote the Thom space of  $\mathcal{N}(i)$ , as in [MV99, Definition 2.16]. By [MV99, Theorem 2.23], there is a canonical isomorphism

$$(5) \quad \text{Th}(\mathcal{N}(i)) \cong \mathbb{A}_k^1 / (\mathbb{A}_k^1 - i(Z))$$

in  $\mathbf{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ . Since  $\mathbb{A}_k^1 - i(Z) \rightarrow \mathbb{A}_k^1$  is an open immersion, it is a monomorphism and therefore a cofibration. It follows that  $\mathbb{A}_k^1 / (\mathbb{A}_k^1 - i(Z))$  is equivalent to the homotopy cofiber of  $\mathbb{A}_k^1 - i(Z) \rightarrow \mathbb{A}_k^1$ . Since  $\mathbb{A}_k^1 \rightarrow *$  is a weak equivalence, this homotopy cofiber is equivalent to the homotopy cofiber of  $\mathbb{A}_k^1 - i(Z) \rightarrow *$ . This later homotopy cofiber is equivalent to the unreduced suspension  $S(\mathbb{A}_k^1 - i(Z))$ .

Let  $\mathcal{O}$  denote the trivial bundle of rank 1 over  $Z$ ,  $\mathcal{O} = Z \times \mathbb{A}^1$ , and let  $\mathbb{P}\mathcal{N}(i) \rightarrow \mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})$  denote the closed embedding at infinity. The vector bundle  $\mathcal{N}(i)$  is trivial of rank



1 over  $Z$ . Since there are no automorphisms of  $\mathbb{A}^1$  fixing 0 and 1, we have a canonical coordinate  $z$  with  $\mathbb{A}^1 = \operatorname{Spec} k[z]$ . For any  $p \in k$ , the map  $k[z]/\langle z - p \rangle \rightarrow \langle z - p \rangle / \langle z - p \rangle^2$  sending  $f(z)$  to  $f(p)(z - p)$  gives a canonical trivialization of the normal bundle of the closed immersion  $\operatorname{Spec} k[z]/\langle z - p \rangle \rightarrow \mathbb{A}^1$ . This gives a trivialization of  $\mathcal{N}(i)$ . We obtain a canonical isomorphism  $\mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i) \cong \mathbb{P}^1 \vee \mathbb{P}^1$ . Use the coordinate  $z$  on  $\mathbb{A}^1$  and  $\mathbb{G}_m = \operatorname{Spec} k[z, \frac{1}{z}]$ . The reasoning above gives an equivalence  $S\mathbb{G}_{m,k} \cong \mathbb{A}^1/\mathbb{G}_{m,k} \rightarrow \mathbb{P}_k^1$ . The natural map  $S\mathbb{G}_{m,k} \rightarrow \Sigma\mathbb{G}_{m,k}$  is a sectionwise weak equivalence, and thus gives a canonical isomorphism in the homotopy category. This yields a canonical isomorphism  $\Sigma(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \cong \mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i)$  in  $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ . By [MV99, Proposition 2.17.3.], there is a canonical equivalence  $\mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i) \rightarrow \operatorname{Th}(\mathcal{N}(i))$ . Combining with (5) produces the desired canonical isomorphism.  $\square$

**Corollary 3.2.** *For any choice of base point of  $\mathbb{P}_k^1 - \{0, 1, \infty\}$ , there is a canonical isomorphism  $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \rightarrow \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$  in  $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ .*

*Proof.* Since the natural map from the unreduced suspension to the reduced is a sectionwise weak equivalence, the unreduced suspension is equivalent to the reduced in a canonical manner. Thus we have a canonical equivalence between  $S(\mathbb{P}_k^1 - \{0, 1, \infty\})$  and  $\Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$ , so the corollary follows from Proposition 3.1.  $\square$

Let  $c_i : \mathbb{G}_m \vee \mathbb{G}_m \rightarrow \mathbb{G}_m$  for  $i = 1$  (respectively  $i = 2$ ) be the map which crushes the first (respectively second) summand of  $\mathbb{G}_m$ . Let  $a_1 : \mathbb{P}_k^1 - \{0, 1, \infty\} \rightarrow \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$  denote the open immersion. Let  $a_2 : \mathbb{P}_k^1 - \{0, 1, \infty\} \cong \operatorname{Spec} k[z, \frac{1}{z}, \frac{1}{z-1}] \rightarrow \mathbb{G}_m \cong \operatorname{Spec} k[z, \frac{1}{z}]$  be given by  $a_2^*(z) = z - 1$ . Consider these maps as unpointed.

**Lemma 3.3.** *Let  $i = 1$  or  $2$ . The following diagram, whose top horizontal map is the isomorphism of Proposition 3.1,*

$$\begin{array}{ccccc}
 & & S\mathbb{G}_m & \xrightarrow{\cong} & \Sigma\mathbb{G}_m \\
 & \nearrow S a_i & & & \nwarrow \Sigma c_i \\
 S(\mathbb{P}_k^1 - \{0, 1, \infty\}) & \xrightarrow{\cong} & \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\}) & \xleftarrow{\quad} & \Sigma(\mathbb{G}_m \vee \mathbb{G}_m)
 \end{array}$$

*is commutative in  $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ .*

*Proof.* We keep the notation of the proof of Proposition 3.1. Let  $i_0 : \{0\} \rightarrow Z$  and  $i_1 : \{1\} \rightarrow Z$  be the closed (and open) immersions, and let  $j_k = i \circ i_\ell$  for  $\ell = 0, 1$ . Let  $\mathcal{N}(j_\ell)$  denote the normal bundle to  $j_\ell$ . The decomposition of  $Z$  as the disjoint union  $Z = \{0\} \amalg \{1\}$  gives a decomposition  $\mathcal{N}(i) = \mathcal{N}(j_0) \amalg \mathcal{N}(j_1)$ . The maps of pairs  $(\mathcal{N}(j_\ell), \mathcal{N}(j_\ell) - 0) \rightarrow (\mathcal{N}, \mathcal{N} - 0)$  for  $\ell = 0, 1$  determine maps  $\operatorname{Th}(\mathcal{N}(j_\ell)) \rightarrow \operatorname{Th}(\mathcal{N}(i))$  which combine to give an isomorphism

$$\operatorname{Th}(\mathcal{N}(j_0)) \vee \operatorname{Th}(\mathcal{N}(j_1)) \rightarrow \operatorname{Th}(\mathcal{N}(i)).$$

Mapping  $\operatorname{Th}(\mathcal{N}(j_0))$  to the basepoint thus determines a map  $\operatorname{Th}(\mathcal{N}(i)) \rightarrow \operatorname{Th}(\mathcal{N}(j_1))$ . And we have the analogous map  $\operatorname{Th}(\mathcal{N}(i)) \rightarrow \operatorname{Th}(\mathcal{N}(j_0))$ .

The diagram

$$\begin{array}{ccc} \mathbb{A}_k^1/(\mathbb{A}_k^1 - \mathbf{i}(Z)) & \longleftarrow & \mathrm{Th}(\mathcal{N}(\mathbf{i})) \\ \downarrow & & \downarrow \\ \mathbb{A}_k^1/(\mathbb{A}_k^1 - \mathbf{j}_0(\{0\})) & \longleftarrow & \mathrm{Th}(\mathcal{N}(\mathbf{j}_\ell)) \end{array}$$

in  $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  is commutative by the functoriality of blow-ups and the construction of the canonical isomorphism of [MV99, Theorem 2.23].

Use the trivialization of  $\mathcal{N}(\mathbf{j}_\ell)$  from the proof of Proposition 3.1. We obtain an isomorphism  $\mathrm{Th}(\mathcal{N}(\mathbf{j}_\ell)) \rightarrow \mathbb{P}^1$ . This isomorphism fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Th}(\mathcal{N}(\mathbf{i})) & \longleftarrow & \mathbb{P}^1 \vee \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathrm{Th}(\mathcal{N}(\mathbf{j}_\ell)) & \longleftarrow & \mathbb{P}^1 \end{array}$$

where the top horizontal map is as in the proof of Proposition 3.1, and the right vertical morphism crushes the factor not corresponding to  $\ell$ .

Place the two previous commutative diagrams side by side and use the isomorphism  $\mathbb{P}^1 \rightarrow \Sigma \mathbb{G}_m$  from the proof of Proposition 3.1 to replace the  $\mathbb{P}^1$ 's with  $\Sigma \mathbb{G}_m$ 's. Then note that the composition

$$\mathbb{A}_k^1/(\mathbb{A}_k^1 - \mathbf{i}(Z)) \rightarrow \mathbb{A}_k^1/(\mathbb{A}_k^1 - \mathbf{j}_\ell(\{\ell\})) \rightarrow \mathbb{P}^1 \rightarrow \Sigma \mathbb{G}_m$$

is the composition of  $\mathbf{Sa}_\ell$  with  $\mathbf{SG}_m \rightarrow \Sigma \mathbb{G}_m$  after identifying  $\mathbb{A}_k^1/(\mathbb{A}_k^1 - \mathbf{i}(Z)) \cong \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$ . This proves the proposition.  $\square$

Let  $\overline{01}$  denote the tangential base point of  $\mathbb{P}_k^1 - \{0, 1, \infty\}$  at  $0$  pointing in the direction of  $1$ , as in [Del89, §15] [Nak99], so  $\overline{01}$  determines the fiber functor associated to the geometric point

$$\mathbb{P}_k^1 - \{0, 1, \infty\} = \mathrm{Spec} \, k[z, \frac{1}{z}, \frac{1}{1-z}] \leftarrow \mathrm{Spec} \, \cup_{n \in \mathbb{Z}_{>0}} \overline{k}((z^{1/n}))$$

$$k[z, \frac{1}{z}, \frac{1}{1-z}] \rightarrow k(z) \rightarrow \cup_{n \in \mathbb{Z}_{>0}} \overline{k}((z^{1/n})).$$

Let  $\pi = \pi_1^{\mathrm{ét}}(\mathbb{P}_k^1 - \{0, 1, \infty\}, \overline{01})$ . Since the étale fundamental group is invariant under algebraically closed base change in characteristic  $0$ , we have a canonical isomorphism  $\pi \cong \pi_1^{\mathrm{ét}}(\mathbb{P}_{\mathbb{C}}^1 - \{0, 1, \infty\}, \overline{01})$ . There is a canonical isomorphism between  $\pi_1^{\mathrm{ét}}(\mathbb{P}_{\mathbb{C}}^1 - \{0, 1, \infty\}, \overline{01})$  and the profinite completion of the topological fundamental group. Let  $\mathbf{x}$  be the element of the topological fundamental group represented by a small counter-clockwise loop around  $0$  based at  $\overline{01}$ , and let  $\mathbf{y}$  be the path formed by traveling along  $[0, 1]$ , then traveling along the image of  $\mathbf{x}$  under  $z \mapsto 1 - z$ , and then traveling back from  $1$  to  $0$  along  $[0, 1]$ . Putting this together, we have fixed an isomorphism

$$\pi \cong \langle \mathbf{x}, \mathbf{y} \rangle^\wedge$$

between  $\pi$  and the profinite completion of the free group on two generators  $\mathbf{x}$  and  $\mathbf{y}$ . Recall that in Example 2.4, we have defined  $\pi' = \langle \mathbf{x}, \mathbf{y} \rangle^\wedge$  and maps out of  $\pi_1(\mathbb{G}_{m, \overline{k}} \vee \mathbb{G}_{m, \overline{k}}, *)$  to inverse systems of finite groups factor through  $\pi'$  by Lemma 2.3.

Let  $x_n^*, y_n^* \in \text{Hom}(\pi, \mathbb{Z}/n)$  be defined by  $x_n^*(x) = 1$ ,  $x_n^*(y) = 0$ ,  $y_n^*(x) = 0$ , and  $y_n^*(y) = 1$ . By Proposition 2.6,  $H^1(\mathbb{P}_k^1 - \{0, 1, \infty\}, \mathbb{Z}/n)$  is a free  $\mathbb{Z}/n$ -module with basis  $\{x_n^*, y_n^*\}$ . Making the analogous definitions of  $x_n^*$  and  $y_n^*$  with  $\pi'$  replacing  $\pi$ , Proposition 2.6 and Lemma 2.3 show that  $H^1(\mathbb{G}_{m, \bar{k}} \vee \mathbb{G}_{m, \bar{k}}, \mathbb{Z}/n)$  is a free  $\mathbb{Z}/n$ -module with basis  $\{x_n^*, y_n^*\}$ . By Proposition 2.5, we obtain isomorphisms  $H^2(\Sigma X_{\bar{k}}, \mathbb{Z}/n) \cong \mathbb{Z}/n x_n^* \oplus \mathbb{Z}/n y_n^*$  for  $X = \mathbb{P}_k^1 - \{0, 1, \infty\}$ , and  $\mathbb{G}_m \vee \mathbb{G}_m$ .

Let  $\wp : \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \rightarrow \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$  be any map determining the canonical isomorphism of Proposition 3.1.

**Proposition 3.4.**  $H^2(\wp_{\bar{k}}, \mathbb{Z}/n)$  is computed by  $H^2(\wp_{\bar{k}}, \mathbb{Z}/n)(x_n^*) = x_n^*$  and  $H^2(\wp_{\bar{k}}, \mathbb{Z}/n)(y_n^*) = y_n^*$ .

*Proof.* By an abuse of notation, let  $\wp$  also denote the composite morphism  $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \rightarrow \Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\}) \rightarrow S(\mathbb{P}_k^1 - \{0, 1, \infty\})$  in  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ , and identify  $H^2(S(\mathbb{P}_k^1 - \{0, 1, \infty\}), \mathbb{Z}/n)$  with  $H^2(\Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\}), \mathbb{Z}/n)$  and  $H^2(S\mathbb{G}_m, \mathbb{Z}/n)$  with  $H^2(\Sigma\mathbb{G}_m, \mathbb{Z}/n)$  by the isomorphisms in the statement of Lemma 3.3. Let  $\Sigma\mathbf{a}_i$  denote the composition of  $\mathbf{S}\mathbf{a}_i$  with the canonical map  $S\mathbb{G}_m \rightarrow \Sigma\mathbb{G}_m$ .

Then Lemma 3.3 says that  $\Sigma\mathbf{a}_i \circ \wp = \Sigma\mathbf{c}_i$ . The dual to the counterclockwise loop based at 1 in  $\mathbb{G}_m(\mathbb{C})$  determines a canonical element  $z_n^*$  of  $H^2(\Sigma\mathbb{G}_{m, \bar{k}}, \mathbb{Z}/n)$  by the comparison between the étale and topological fundamental groups [SGAI, XII Corollaire 5.2], Proposition 2.5, and Proposition 2.6. By the construction of  $x_n^*$  and  $y_n^*$ , we have that  $(\Sigma\mathbf{a}_1)^*(z_n^*) = x_n^*$ ,  $(\Sigma\mathbf{a}_2)^*(z_n^*) = y_n^*$ ,  $(\Sigma\mathbf{c}_1)^*(z_n^*) = x_n^*$  and  $(\Sigma\mathbf{c}_2)^*(z_n^*) = y_n^*$ . This shows the proposition.  $\square$

#### 4. DESUSPENDING $\Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$

We use the Galois action on  $\pi_1^{\text{ét}}(\mathbb{P}_k^1 - \{0, 1, \infty\})$  to show that  $\mathbb{P}_k^1 - \{0, 1, \infty\}$  and  $\mathbb{G}_{m, k} \vee \mathbb{G}_{m, k}$  are distinct desuspensions of  $\Sigma(\mathbb{P}_k^1 - \{0, 1, \infty\})$ . Recall the definition of  $\pi'$  from Example 2.2. Here are the needed facts about the Galois action on  $\pi = \pi_1^{\text{ét}}(\mathbb{P}_k^1 - \{0, 1, \infty\}, \overline{01})$ .

An element  $g \in G$  acts on  $\pi$  by

$$(6) \quad g(x) = x^{x(g)} \quad g(y) = f(g)^{-1} y^{x(g)} f(g)$$

where  $f : G \rightarrow [\pi]_2$  is a cocycle with values in the commutator subgroup  $[\pi]_2$  of  $\pi$ . See [Iha94, Proposition 1.6]. Since  $\overline{01}$  is a rational tangential base-point,  $\overline{01}$  splits the homomorphism  $\pi_1^{\text{ét}}(\mathbb{P}_k^1 - \{0, 1, \infty\}, \overline{01}) \rightarrow \pi_1^{\text{ét}} \text{Spec } k \cong G$ , giving an isomorphism  $\pi_1^{\text{ét}}(\mathbb{P}_k^1 - \{0, 1, \infty\}, \overline{01}) \cong \pi \rtimes G$ .

Let  $\pi = [\pi]_1 \supseteq [\pi]_2 \supseteq [\pi]_3 \supseteq \dots$  denote the lower central series of  $\pi$ , so  $[\pi]_n$  is the closure of the subgroup generated by commutators of elements of  $\pi$  with elements of  $[\pi]_{n-1}$ . Use the analogous notation for the lower central series of any profinite group.

Let  $\iota : \pi' \rightarrow \pi$  be the homomorphism of groups  $\iota(x) = x$  and  $\iota(y) = y$ , and let  $\iota^{\text{ab}} : (\pi')^{\text{ab}} \rightarrow \pi^{\text{ab}}$  denote the induced map on abelianizations. Note that  $\iota^{\text{ab}}$  is  $G$ -equivariant.

**Lemma 4.1.** *Let  $k$  be a number field not containing the square root of 2. Then there is no continuous homomorphism  $\pi' \times G \rightarrow \pi \rtimes G$  over  $G$  inducing  $\iota^{\text{ab}} \rtimes 1_G$  after abelianization.*

*Proof.* Suppose to the contrary that  $\theta$  is such a map. Because the subgroups of the lower central series are characteristic,  $\theta$  induces a commutative diagram

$$(7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & [\pi]_n/[\pi]_{n+1} & \longrightarrow & \pi/[\pi]_{n+1} \rtimes G & \longrightarrow & \pi/[\pi]_n \rtimes G \longrightarrow 1 \\ & & \bar{\theta}_n^{n+1} \uparrow & & \bar{\theta}_{n+1} \uparrow & & \bar{\theta}_n \uparrow \\ 1 & \longrightarrow & [\pi']_n/[\pi']_{n+1} & \longrightarrow & \pi'/[\pi']_{n+1} \rtimes G & \longrightarrow & \pi'/[\pi']_n \rtimes G \longrightarrow 1 \end{array}$$

Thus if  $\bar{\theta}_n^{n+1}$  and  $\bar{\theta}_n$  are isomorphisms, so is  $\bar{\theta}_{n+1}$ . Since  $\pi'$  and  $\pi$  are isomorphic to the profinite completion of the free group on two generators,  $[\pi']_n/[\pi']_{n+1}$  and  $[\pi]_n/[\pi]_{n+1}$  are isomorphic to the degree  $n$  graded component of the free Lie algebra on the same generators over  $\mathbb{Z}^\wedge$ . Since  $\bar{\theta}_2 = \iota^{ab} \rtimes 1_G$  is an isomorphism, it follows that  $\bar{\theta}_n^{n+1}$  is an isomorphism. By induction, it follows that  $\bar{\theta}_n$  is an isomorphism for all  $n$ .

The extension

$$1 \rightarrow [\pi]_n/[\pi]_{n+1} \rightarrow \pi/[\pi]_{n+1} \rtimes G \rightarrow \pi/[\pi]_n \rtimes G \rightarrow 1$$

is classified by the element of  $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$  represented by the inhomogeneous cocycle  $\varphi_n$

$$\varphi_n(\gamma \rtimes g, \eta \rtimes h) = s(\gamma)gs(\eta)s(\gamma g\eta)^{-1}$$

where  $s : \pi/[\pi]_n \rightarrow \pi/[\pi]_{n+1}$  is a continuous set-theoretic section of the quotient map  $\pi/[\pi]_{n+1} \rightarrow \pi/[\pi]_n$ . See for example [Bro94, IV 3]. Let  $\varphi'_n$  denote the analogous inhomogeneous cocycle obtained by replacing  $\pi$  with  $\pi'$ .

This association of a class in  $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$  to an extension of  $\pi/[\pi]_n \rtimes G$  by  $[\pi]_n/[\pi]_{n+1}$  induces a bijection between  $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$  and isomorphism classes of extensions [Bro94, IV Theorem 3.12]. Since  $\bar{\theta}_n$  is an isomorphism, it follows that  $\varphi'_n$  and  $(\bar{\theta}_n)^*\varphi_n$  represent the same class in  $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$ .

By (6) and (3),

$$\begin{aligned} \pi/[\pi]_2 &\cong \mathbb{Z}^\wedge(1)x \oplus \mathbb{Z}^\wedge(1)y \\ [\pi]_2/[\pi]_3 &\cong \mathbb{Z}^\wedge(2)[x, y] \end{aligned}$$

$$(8) \quad [\pi]_3/[\pi]_4 \cong \mathbb{Z}^\wedge(3)[[x, y], x] \oplus \mathbb{Z}^\wedge(3)[[x, y], y],$$

and the same isomorphisms hold with  $\pi'$  replacing  $\pi$ .

We claim that  $\bar{\theta}_3$  is given by

$$(9) \quad \bar{\theta}_3(x \rtimes 1) = x \rtimes 1 \quad \bar{\theta}_3(y \rtimes 1) = y \rtimes 1 \quad \bar{\theta}_3(1 \rtimes g) = [x, y]^{c_1(g)} \rtimes g$$

for all  $g \in G$ , where

$$c : G \rightarrow \mathbb{Z}^\wedge(2)$$

is a cocycle. To see this, note that the hypothesis on  $\bar{\theta}_2$  implies that  $\bar{\theta}_3(x \rtimes 1) = x[x, y]^{c_1(g)} \rtimes 1$  with  $c_1(g)$  in  $\mathbb{Z}^\wedge$ . Similarly,  $\bar{\theta}_3(1 \rtimes g) = [x, y]^{c(g)} \rtimes g$  with  $c(g)$  in  $\mathbb{Z}^\wedge$ . Since  $\theta$  is a homomorphism, we have  $\bar{\theta}_3(gx) = \bar{\theta}_3(g)\bar{\theta}_3(x)$ . Since  $gx = x^{x(g)} \rtimes g$  and  $[x, y]$  is in the center, we have

$$\bar{\theta}_3(gx) = \bar{\theta}_3(x)^{x(g)}\bar{\theta}_3(g) = x^{x(g)}[x, y]^{c_1(g)x(g)+c(g)} \rtimes g.$$

On the other hand,

$$\bar{\theta}_3(g)\bar{\theta}_3(x) = ([x, y]^{c(g)} \rtimes g)(x[x, y]^{c_1(g)} \rtimes 1) = x^{\chi(g)}[x, y]^{c_1(g)\chi(g)^2+c(g)} \rtimes g.$$

Thus  $c_1(g)\chi(g)^2 + c(g) = c_1(g)\chi(g) + c(g)$  for all  $g$  in  $G$ . It follows that  $c_1(g) = 0$ . Since  $f(g)$  is in  $[\pi]_2$  and elements of  $[\pi]_2$  are all in the center of  $\pi/[\pi]_3$ , switching  $x$  and  $y$  induces an isomorphism on  $\pi/[\pi]_3$ . The same argument therefore implies that  $\bar{\theta}_3(y) = y$ . Since  $\bar{\theta}_3$  is a homomorphism when restricted to  $1 \rtimes G$ , it follows that  $c$  is a cocycle, showing (9).

By (8), we have a direct sum decomposition

$$H^2(\pi/[\pi]_3 \rtimes G, [\pi]_3/[\pi]_4) \cong H^2(\pi/[\pi]_3 \rtimes G, \mathbb{Z}^\wedge(3))[[x, y], x] \oplus H^2(\pi/[\pi]_3 \rtimes G, \mathbb{Z}^\wedge(3))[[x, y], y].$$

Define  $\varphi_{3,[[x,y],x]}$  and  $\varphi_{3,[[x,y],y]}$  so that under this isomorphism  $\varphi_3$  decomposes as  $\varphi_3 = \varphi_{3,[[x,y],x]} \oplus \varphi_{3,[[x,y],y]}$ . It was calculated in [Wic12] that  $\varphi_{3,[[x,y],x]}$  is represented by the cocycle mapping  $(y^{a_1}x^{b_1}[x, y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x, y]^{c_2} \rtimes g_2)$  to

$$c_1\chi(g_1)b_2 + \binom{b_1+1}{2}\chi(g_1)a_2 + b_1\chi(g_1)^2a_2b_2 - \frac{\chi(g_1)-1}{2}\chi(g_1)^2c_2$$

and that  $\varphi_{3,[[x,y],y]}$  is represented by the cocycle mapping  $(y^{a_1}x^{b_1}[x, y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x, y]^{c_2} \rtimes g_2)$  to

$$c_1\chi(g_1)a_2 + b_1\binom{\chi(g_1)a_2+1}{2} - \chi(g_1)\binom{\chi(g_1)}{2}c_2 - f(g_1)\chi(g_1)a_2$$

where  $f : G \rightarrow \mathbb{Z}^\wedge(2)$  is such that  $f(g) = [x, y]^{f(g)}$  in  $\pi/[\pi]_3$ .

We may similarly decompose  $\varphi'_3$  as  $\varphi'_3 = \varphi'_{3,[[x,y],x]} \oplus \varphi'_{3,[[x,y],y]}$ . By the above calculation of  $\bar{\theta}_3$ , and the expressions (6) and (3) for the  $G$ -action on  $\pi$  and  $\pi'$ , we have that  $\varphi'_{3,[[x,y],x]}$  and  $\varphi'_{3,[[x,y],y]}$  are obtained from the expressions for  $\varphi_{3,[[x,y],x]}$  and  $\varphi_{3,[[x,y],y]}$  by setting  $f = 0$ .

It follows that  $\varphi'_3 - (\bar{\theta}_3)^*\varphi_3$  is represented by the direct sum of two cocycles, given by sending  $(y^{a_1}x^{b_1}[x, y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x, y]^{c_2} \rtimes g_2)$  to

$$(-c(g_1)\chi(g_1)b_2 + \frac{\chi(g_1)-1}{2}\chi(g_1)^2c(g_2))[[x, y], x]$$

and

$$(-c(g_1)\chi(g_1)a_2 + \chi(g_1)\binom{\chi(g_1)}{2}c(g_2) + f(g_1)\chi(g_1)a_2)[[x, y], y]$$

respectively.

Using the above direct sum decomposition of  $H^2(\pi/[\pi]_3 \rtimes G, [\pi]_3/[\pi]_4)$ , this implies that

$$\varphi'_{3,[[x,y],x]} - (\bar{\theta}_3)^*\varphi_{3,[[x,y],x]} = -c \cup b + \frac{\chi(g)-1}{2} \cup c$$

and

$$\varphi'_{3,[[x,y],y]} - (\bar{\theta}_3)^*\varphi_{3,[[x,y],y]} = -c \cup a + \frac{\chi(g)-1}{2} \cup c + f \cup a,$$

where these equalities are in  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$ , and where  $f : G \rightarrow \mathbb{Z}^\wedge(2)$  is considered via pullback as an element of  $H^1(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(2))$ ,  $a : \pi'/[\pi']_3 \rtimes G \rightarrow \mathbb{Z}^\wedge(1)$  is the cocycle

$y^a x^b [x, y]^c \rtimes g \mapsto a, b$  is defined similarly, and  $\frac{x(g)-1}{2}$  is the cocycle  $g \mapsto \frac{x(g)-1}{2}$  taking values in  $\mathbb{Z}^\wedge(1)$  pulled back to  $\pi'/[\pi']_3 \rtimes G$ .

As shown above, the existence of  $\theta$  therefore implies that  $-c \cup b + \frac{x(g)-1}{2} \cup c = 0$  and  $-c \cup a + \frac{x(g)-1}{2} \cup c + f \cup a = 0$  in  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$ .

Consider first the equality  $-c \cup b + \frac{x(g)-1}{2} \cup c = 0$ . Since the cup product is graded-commutative, we may rewrite this equality as  $(b + \frac{x(g)-1}{2}) \cup c = 0$ . The quotient map  $\mathbb{Z}^\wedge(3) \rightarrow \mathbb{Z}/2$  determines map  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3)) \rightarrow H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}/2)$ . Passing to the image under this map, we have an equality  $(b + \frac{x(g)-1}{2}) \cup \bar{c} = 0$ , where  $\bar{c}$  denotes the image of  $c$  and  $(b + \frac{x(g)-1}{2})$  denotes the image  $b + \frac{x(g)-1}{2}$ . Recall that for any  $\beta \in k^*$  with a chosen compatible system of  $n$ th roots of  $\beta$ , there is a Kummer cocycle  $\kappa(\beta) : G \rightarrow \hat{\mathbb{Z}}(1) \cong \varprojlim_n \mu_n(\bar{k})$  defined by  $g \sqrt[n]{\beta} = \kappa(b)(g)_n \sqrt[n]{\beta}$  where  $\kappa(b)(g)_n$  is the element of  $\mu_n(\bar{k})$  determined by  $\kappa(b)(g)$ . We may define a homomorphism  $G \rightarrow \pi'/[\pi']_3 \rtimes G$  by  $g \mapsto y^{\kappa(\beta)} \rtimes g$ . Pulling back the equality  $(b + \frac{x(g)-1}{2}) \cup \bar{c} = 0$  by this homomorphism gives the equality  $(\kappa(b) + \frac{x(g)-1}{2}) \cup \bar{c} = 0$  in  $H^2(G, \mathbb{Z}/2)$  because  $c$  and  $\frac{x(g)-1}{2}$  are pulled back from  $G$ . Note that any element of  $H^1(G, \mathbb{Z}/2)$  is of the form  $(\kappa(b) + \frac{x(g)-1}{2})$  for an appropriate choice of  $\beta$ . By the non-degeneracy of the cup product  $H^1(G, \mathbb{Z}/2) \otimes H^1(G, \mathbb{Z}/2) \rightarrow H^2(G, \mathbb{Z}/2)$ , it follows that  $\bar{c} \in H^2(G, \mathbb{Z}/2)$  is zero.

Consider now the second equality  $-c \cup a + \frac{x(g)-1}{2} \cup c + f \cup a = 0$  in  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$ , and again pass to the image under  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3)) \rightarrow H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}/2)$ . Since  $\bar{c} = 0$  in  $H^2(G, \mathbb{Z}/2)$ , we have  $\bar{f} \cup \bar{a} = 0$  in  $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$ . On the other hand,  $f : G \rightarrow \mathbb{Z}^\wedge(2)$  is known by work of Ihara [Iha91, 6.3 Thm p.115] [IKY87], Anderson [And89], and Coleman [Col89], and we can show that this is inconsistent with  $\bar{f} \cup \bar{a} = 0$  in the following manner. Namely,  $f(g) = \frac{1}{24}(\chi(g)^2 - 1)$ . See [Wic12, 12.5.2]. By [Wic12, Lemma 31], the image of  $f$  under the map  $H^1(G, \mathbb{Z}^\wedge(2)) \rightarrow H^1(G, \mathbb{Z}/2) \cong k^*/(k^*)^2$  is represented by  $2 \in k^*$ . For any  $\alpha \in k^*$  we may choose a compatible system of  $n$ th roots of  $\alpha$  and define a homomorphism  $G \rightarrow \pi'/[\pi']_3 \rtimes G$  by  $g \mapsto y^{\kappa(\alpha)} \rtimes g$ . Pulling back  $\bar{f} \cup \bar{a}$  by this homomorphism gives  $\bar{f} \cup \kappa(\alpha) \in H^2(G, \mathbb{Z}/2)$ . Thus  $\kappa(2) \cup \kappa(\alpha) = 0 \in H^2(G, \mathbb{Z}/2)$  for all  $\alpha \in k^*$ . Since  $k$  does not contain the square root of 2, this contradicts the nondegeneracy of the the cup product, giving the desired contradiction.  $\square$

**Theorem 4.2.** *Let  $k$  be a finite extension of  $\mathbb{Q}$  not containing the square root of 2. There is no morphism  $\rho : \mathbb{G}_{m,k} \vee \mathbb{G}_{m,k} \rightarrow \mathbb{P}_k^1 - \{0, 1, \infty\}$  in  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$  such that  $\Sigma\rho = \wp$  in  $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ .*

*Proof.* Suppose to the contrary that we have such a morphism  $\rho$ . The geometric point  $\overline{01}$  of  $\mathbb{P}_k^1 - \{0, 1, \infty\}$  and the extension of the  $k$ -basepoint of  $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$  to a geometric point allow us to consider  $(\mathbb{P}_k^1 - \{0, 1, \infty\}, \overline{01})$  and  $(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *)$  as objects of  $\mathbf{sPre}(\mathbf{Sm}_k)^+$ . Since  $\mathbb{P}_k^1 - \{0, 1, \infty\}$  and  $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$  have connected étale homotopy type,  $\rho$  is a morphism in  $\mathbf{sPre}(\mathbf{Sm}_k)_c^+$ . Thus  $\rho$  induces an outer continuous homomorphism  $\rho_* : \pi' \rtimes G \rightarrow \pi \rtimes G$  by



Proposition 2.1, Lemma 2.3, and taking the inverse limit. We may choose a continuous homomorphism over  $G$  representing  $\rho_*$ . By a slight abuse of notation, we call this representative  $\rho_*$  as well.

Let  $(\rho_{\bar{k}})_*$  denote the induced map  $\pi' \rightarrow \pi$ . It follows from Proposition 2.6 that the induced map  $\rho^* : H^1(\mathbb{P}_{\bar{k}}^1 - \{0, 1, \infty\}, \mathbb{Z}/n) \rightarrow H^1(\mathbb{G}_{m, \bar{k}} \vee \mathbb{G}_{m, \bar{k}}, \mathbb{Z}/n)$  is computed  $\rho^* = \text{Hom}((\rho_{\bar{k}})_*, \mathbb{Z}/n)$ . By Proposition 2.5,  $H^2(\wp_{\bar{k}}) = H^1(\rho_{\bar{k}})$ . Combining the two previous, we have  $H^2(\wp_{\bar{k}}) = \text{Hom}((\rho_{\bar{k}})_*, \mathbb{Z}/n)$ . By Proposition 3.4, it follows that  $\text{Hom}(\rho_*, \mathbb{Z}/n)(\chi_n^*) = \chi_n^*$  and  $\text{Hom}(\rho_*, \mathbb{Z}/n)(y_n^*) = y_n^*$ . Since  $n$  is arbitrary, it follows that  $(\rho_{\bar{k}})_*^{\text{ab}} = \iota^{\text{ab}}$ .

We claim that after modifying  $\rho_*$  by an inner automorphism, the map  $\rho_*^{\text{ab}} : \pi'/[\pi']_2 \rtimes G \rightarrow \pi/[\pi]_2 \rtimes G$  induced by  $\rho_*$  is  $\iota^{\text{ab}} \rtimes 1_G$ . Note the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi/[\pi]_2 & \longrightarrow & \pi/[\pi]_2 \rtimes G & \longrightarrow & G \longrightarrow 1 \\ & & \uparrow (\rho_{\bar{k}})_*^{\text{ab}} & & \uparrow \rho_*^{\text{ab}} & & \uparrow 1_G \\ 1 & \longrightarrow & \pi'/[\pi']_2 & \longrightarrow & \pi'/[\pi']_2 \rtimes G & \longrightarrow & G \longrightarrow 1. \end{array}$$

Since  $1_G$  and  $(\rho_{\bar{k}})_*^{\text{ab}}$  are isomorphisms, so is  $\rho_*^{\text{ab}}$ . It follows by induction that  $\overline{(\rho_*)}_n : \pi'/[\pi']_n \rtimes G \rightarrow \pi/[\pi]_n \rtimes G$  is an isomorphism, cf. (7).

Let  $\varphi_2 \in H^2(\pi/[\pi]_2 \rtimes G, [\pi]_2/[\pi]_3)$  be the element classifying

$$1 \rightarrow [\pi]_2/[\pi]_3 \rightarrow \pi/[\pi]_3 \rtimes G \rightarrow \pi/[\pi]_2 \rtimes G \rightarrow 1,$$

and define  $\varphi'_2$  by replacing  $\pi$  with  $\pi'$  in the definition of  $\varphi_2$ . Since  $\overline{(\rho_*)}_3$  is an isomorphism,  $\overline{(\rho_*)}_2^* \varphi_2 = \varphi'_2$ . By [Wic12, Proposition 7],  $\varphi_2 = \mathbf{b} \cup \mathbf{a}$ , where  $\mathbf{b} : \pi/[\pi]_2 \rtimes G \rightarrow \mathbb{Z}^\wedge(1)$  is the cocycle  $y^a x^b \rtimes g \mapsto \mathbf{b}$  and  $\mathbf{a} : \pi/[\pi]_2 \rtimes G \rightarrow \mathbb{Z}^\wedge(1)$  is the cocycle  $y^a x^b \rtimes g \mapsto \mathbf{a}$ . Since conjugation by  $f(g)$  is trivial in  $\pi/[\pi]_3$ , it follows that  $\varphi'_2 = \mathbf{b} \cup \mathbf{a}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are defined by replacing  $\pi'$  with  $\pi$  in the previous. Because  $(\rho_{\bar{k}})_*^{\text{ab}} = \iota^{\text{ab}}$ , we have  $\overline{(\rho_*)}_2(y^a x^b \rtimes g) = y^{a+\alpha(g)} x^{b+\beta(g)} \rtimes g$  where  $\alpha, \beta : G \rightarrow \mathbb{Z}^\wedge(1)$  are cocycles. Thus  $\overline{(\rho_*)}_2^* \varphi_2 = (\mathbf{b} + \beta) \cup (\mathbf{a} + \alpha)$ . Thus  $(\mathbf{b} + \beta) \cup (\mathbf{a} + \alpha) = \mathbf{b} \cup \mathbf{a}$ . Since the cup product is non-degenerate, it follows that  $\beta$  and  $\alpha$  are trivial in cohomology. Thus after modifying  $\rho$  by an inner automorphism, we may assume  $\rho_*^{\text{ab}} = \iota^{\text{ab}} \rtimes 1_G$ . This contradicts Lemma 4.1.  $\square$

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